Rapid Note

An evolution theory in finite size systems

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Abstract. A new model of evolution is presented for finite size systems. Conditions under which a minority species can emerge, spread and stabilize to a macroscopic size are studied. It is found that space organization is instrumental in addition to a qualitative advantage. Some peculiar topologies ensure the overcome of the initial majority species. However the probability of such local clusters is very small and depend strongly on the system size. A probabilistic phase diagram is obtained for small sizes. It reduces to a trivial situation in the thermodynamic limit, thus indicating the importance of dealing with finite systems in evolution problems. Results are discussed with respect to both Darwin and punctuated equilibria theories.

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This paper provides a simple model to describe competition between two species over a territory and demonstrates the essential role played by finite size systems. First, we show that our model can be described in terms of a probabilistic phase diagram which reduces to a trivial situation when the system size goes to infinity. Second, our model yields a new interpretation of the phenomena of punctuated equilibria in the framework of species evolution.

A possible conclusion is that some socio-economical systems may be characterized by a strong sensitivity to system size. For instance, the macroscopic behavior may change dramatically whether the system is just large or if it is really huge.

The reason of this peculiar property is the existence, in such systems, of statistically very rare configurations which drive the evolution in a new and atypical way. The observation that rare events can develop and reach a macroscopic size has already been noticed in other contexts. Examples are given by the generalized prisoner dilemma problems [1–3] or the recent work by Solomon [4]. Percolation problems give another example where a qualitative change of behavior is observed in the limit of an infinite system [5].

We propose to consider the effect of the system size in the model of competing standards or opinion forming introduced in [6]. There, we have analyzed several properties

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of the model for large 2d systems but without paying attention to the effects that will take place if the system become really huge. Of course, in 2d, the system sizes that are necessary to exhibit these effects are numerically out of reach. Thus, here, we restrict our study to the 1d case which is much more tractable while still having the same generic features as the 2d system.

We start from two populations A and B which are influencing each other or competing for some unique resources. The individuals move in space according to a cellular automata dynamics, similar to the reaction-diffusion model proposed by Chopard and Droz [7]. However, here, we consider only one type of particle with two possible internal states (± 1), coding for the A or B species, respectively.

The model is as follows. Individuals move on a regular, 1d lattice. At each site, there are always four individuals (any combination of A's and B's is possible). Initially, populations A and B are randomly distributed (independently, identically distributed) over the lattice sites, with respective concentrations a_0 and $b_0 = 1 - a_0$.

The evolution time is discretized. At each time step and at each site, two particles randomly chosen among the four travel to the left while the two others travel to the right. Thus, at each time step, the individuals alway move and there are alway four individuals at each lattice site.

After the motion phase, the individuals interact locally with probability k. The interaction takes place in the form of "fights" between the four individuals meeting on the
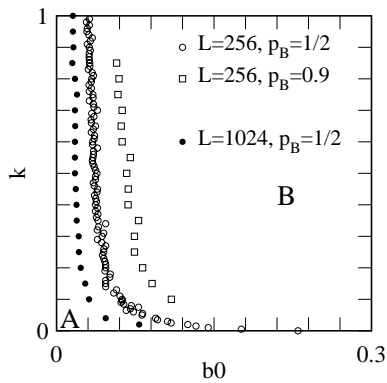
Fig. 1. Probabilistic stationary state phase diagram for systems of size $L = 256$ and $L = 1024$. Contour lines for $p_B = 0.5$ and/or 0.9 are shown. The region marked B indicates that p_B is large whereas it is small in region A.

Fig. 2. Critical size r of a single B cluster that invades the system with probability 0.9, as a function of the aggressiveness k. Dots are the results of the CA model and the solid line is an empirical fit: $k = 1/(r^{1.8})$.

same site. At each fight, the group nature (A or B) is updated according the following biased majority rule:

$$nA + mB \rightarrow \begin{cases} (n+m)A \text{ if } n > m \\ (n+m)B \text{ if } n \leq m \end{cases}$$

where $n + m = 4$. This bias accounts for some advantage (or extra fitness) of species B over A.

The above rule is applied with probability k. Thus, with probability $1 - k$ the group composition does not change since no fight occurs. In between fights both population agents perform a random walk on the lattice, as explained above.

It is clear that the model richness comes from the even confrontations. If only odd fights would happen, the initial majority population would always win within short time intervals.

The key parameters are (i) k, the aggressiveness (probability of confrontation), and (ii) b_0, the initial density of B. The strategy according to which a small minority of B's (with yet a technical, genetic, persuasive advantage) could win against a large population of A's is not obvious. Should they fight very often, try to spread or accept a peace agreement?

In the limit of low aggressiveness ($k \rightarrow 0$), the particles move a long time before fighting. Due to the diffusive motion, correlations between successive fights are destroyed. This is the mean-field level of this dynamical model which corresponds to the theoretical calculations made by Galam in his election model [8]. It was shown that B always wins provided that $b_0 > 0.23$. This limit is visible in Figure 1, for the curve shown with empty circles, for which we have studied numerically the case $k \rightarrow 0$.

As soon as k is not too small local correlations become non negligible and our cellular automata model account for their effects.

Here we study systems of linear size L with periodic boundary conditions. For given values of b_0 and k the dynamics is iterated until a stationary state (either all A or all B) is reached. The interesting point is that the out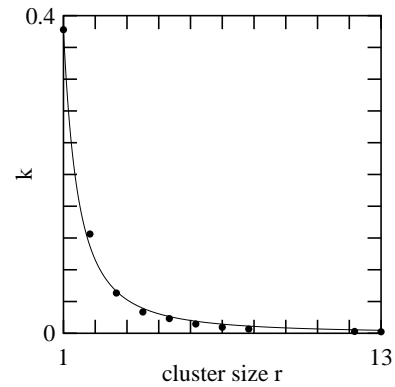come of this experiment is found to be probabilistic: the final state is all B with probability p_B and all A with probability $1 - p_B$. Also, the value of p_B depends crucially on the system size L. As we shall see, when $L \rightarrow \infty$, p_B is one for the all (b_0, k) plane.

For this reason, a standard phase diagram cannot describe the situation properly. Thus, we propose a description in terms of what we call a probabilistic phase diagram: each point of the (b_0, k) plane is assigned a probability p_B that the final state is entirely B. Ideally, this diagram should be represented as a 3D plot. Instead, in Figure 1, we show contour lines corresponding to given probabilities p_B. Note that for the same value of p_B, the isoline is shifted to the left as the system size increases.

These data show that if the aggressiveness k is large enough, initial configurations with a quite low density of B's are able to overcome the large initial majority of the A species. The reason being the presence of B actors which are organized in small clusters such that the diffusion is not effective enough to destroy them. They expand at a rate which makes them win systematically in the fights against A actors. Figure 2 is obtained by considering an initial B cluster of size r in a sea of A's. The plot shows, for each value of k the critical value of r which ensures that the B cluster will invade all the system with probability 0.9.

The result of Figure 2 is independent of the system size L and the question is then how often such clusters appear by chance. In a finite size system, with a given random concentration b_0 of B actors, there is always a finite probability for such small clusters to exist in the initial configuration. When this is the case the system will reach a pure B stationary state. The larger the L the more likely it is to observe such a devastating cluster.

The way the separation line in Figure 1 depends on L has been investigated in Figure 3. The plot shows the location of the transition line as a function of L for a fixed probability $p_B = 1/2$ and different values of k.

One sees that when L increases, the probabilistic line corresponding to a given probability p_B moves to the left and an extrapolation to an infinite size system leads to a collapse of the transition line with the vertical axis
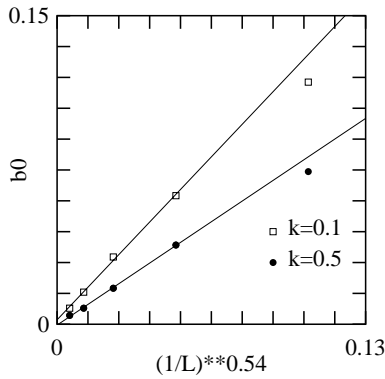
Fig. 3. Dependence of the critical density b_0 of B particle as a function of the system size L, for a wining probability $p_B = 0.5$ and two values of k. We see that the $A - B$ separation line moves as $1/(L^{0.54})$.
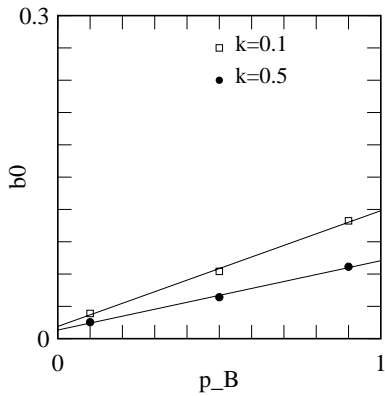

Fig. 4. Critical initial density b_0 as a function of B's probability to win, p_B, for two values of k and $L = 256$. From the assumption of a linear dependence, the value of b_0 for $p_B = 1$ can be interpolated.

for all values $k \neq 0$. For $k = 0$, one recovers the mean-field transition point $b_{0c} = 0.23$, for all values of $p_B > 0$. This is shown in Figure 1 for the case $L = 256$, $p_B = 1/2$, and can be confirmed by direct numerical simulations at $k = 0$ (complete mixing of the individual at each time step).

These results show that the respective behaviors of finite size and infinite size systems are qualitatively different.

Figure 4 shows, for a fixed system size $L = 256$, how the critical density b_0 varies with p_B. For two values of k, the plot suggests an almost linear dependence.

We now discuss in more detail the question of the appearance of the devastating B clusters. More precisely, we would like to know what is the probability $P_L^{(r)}$ to find at least one cluster formed of r consecutive B particles in a system of size L providing that the sites are randomly filled respectively with B particles with probability b_0 and with A particles with probability $a_0 = 1 - b_0$.

This is a difficult problem for arbitrary values of b_0. However, the case $b_0 = 1/2$ is simpler, and a careful
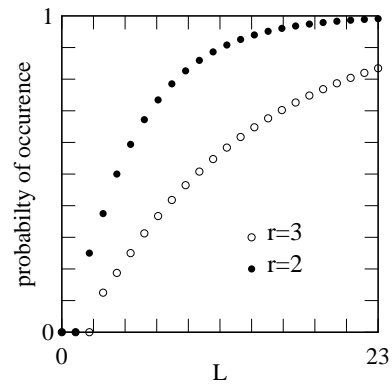

Fig. 5. Probability $P_L^{(r)}$ of finding at least one cluster formed of r consecutive B particles in a system of size L, for $r = 2$ and $r = 3$. The B's are uniformly distributed, with probability $1/2$.

bookkeeping, leads to the following recursion relation:

$$P_L^{(r)} = P_{(L-1)}^{(r)} + a_L^{(r)} \left(\frac{1}{2} \right)^L \tag{1}$$

where the a_L^r are generalized Fibonacci numbers defined by the following recursion relation:

$$a_1^{(r)} = a_2^{(r)} = ... = a_{r-1}^{(r)} = 1,$$
$$a_n^{(r)} = a_{n-1}^{(r)} + a_{n-2}^{(r)} + ... + a_{n-r}^{(r)}, \quad n \leq r + 1 \tag{2}$$

the particular case $r = 2$ corresponds to the usual Fibonacci numbers. The behavior of $P_L^{(r)}$ is shown in Figure 5 for several values of r.

One sees that for a fixed value of r and as L gets large, $P_L^{(r)} \to 1$.

Another interesting problem, related to mutation dynamics, can be investigated within this very simple model. Let us consider a system of size L in a pure A state. Then, due to some mutation mechanism, at each iteration time, beside the former rule, a small randomly chosen fraction p_m of the A population turns to B. As a result of this new dynamics, the A population can be extinct after some time T_{death}. This is due again to the random appearance of small B clusters which are spatially organized in a peculiar topology which eventually overcomes the whole A population.

For a fixed value of p_m this extinction time varies a lot from sample to sample. One can then study the frequency of a given extinction time by performing a large number of different simulations for a given choice of k, L and p_m. Typical results are given in Figure 6.

According to our model, the probability that A survives during T units of time is $P(T_{\text{death}} = T) \leq p(1 - p)^{T-1}$, where p is the probability that a devastating B cluster appears due to a mutation. Such a fit is shown in Figure 6 as a dashed line. Note that, for large T, a power law fit is also possible, as shown by the straight solid line: $P(T_{\text{death}}) \sim T_{\text{death}}^{-2.03}$. However, the first fit is clearly more convincing and may thus provide a new interpretation of the phenomena of punctuated equilibria.
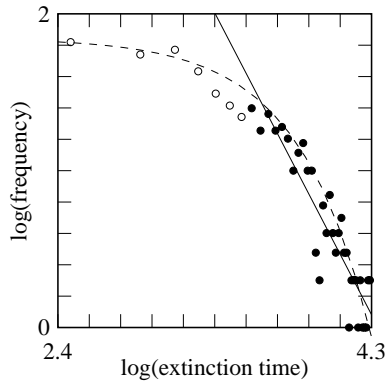
Fig. 6. Frequency distribution of the extinction times for a system of size $L = 256$, for $k = 0.5$ and $p_m = 0.0004$. The distribution is built by organizing the extinction times in 40 bins of size 500. The straight line is a fit of the data shown as black dots and indicates a power law behavior with exponent -2.03. The dashed line is a fit $\gamma(1-p)^{T_{\text{death}}}$, with $\gamma = 70$ and $p = 0.00022$.

In this paper, we have shown how local topology could turn instrumental to the growth of a new species B at the expense of a former one A. The accidental occurrence of these small clusters act as nucleus from which the B's can develop. However, when k is too small, diffusion always destroys such a local organization. Moreover, the larger the system is, the more likely is the occurrence of such rare clusters stressing the qualitative difference between finite and infinite systems when describing evolution. A new interpretation of punctuated equilibrium can then be obtained.

In conclusion, our model embodies some basic features of species dynamics. Its generality makes it applicable to a large range of systems in various different fields like, group opinion forming, economic standard formation, emergence of innovation, and evolution theory.

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